This establishes the identity of the two methods. For e_3^2 according to the ordinary conception of the eccentricity of a conic passes through all real values between zero and infinity. When it becomes negative, it is then the square of what I have elsewhere called the conjugate eccentricity and denoted by e_3'. Hence, in MacCullagh's modular method, if the square of the modulus be conceived to pass through the complete cycle of real values, and our conception enlarged to include imaginary, as well as real, planes of circular section; then Salmon's method appears as a particular case of MacCullagh's; viz.: when e_3^2 is negative. It must be remembered that when e_3^2 changes to $e_3'^2$, θ_1 becomes θ_2, and the planes of real circular section revolve through 90°, so that in (5) y and z should be interchanged.

By comparing equation (2) and (4), it may be argued in the same way that MacCullagh's method is only a particular case of Salmon's. Thus they are mutually inclusive, and there is no reason for regarding one more general than the other.

SOLUTIONS OF EXERCISES.

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SHOW that

$$\sin\theta > \theta - \frac{\theta^3}{3!} + \frac{1}{45}\left[\frac{\theta^5}{2^2} - \frac{\theta^7}{2^9} + \ldots (-)^{m+1}\frac{\theta^{2m+3}}{2^{\frac{1}{2}(m^2+9m+6)}} \pm \ldots\right];$$

the general term being the mth within the brackets. [*W. H. Echols.*]

SOLUTION.

THE general term as stated in the exercise is wrong; it should be

$$(-1)^{n+1}\frac{\theta^{2n-1}}{(2^2-1)(2^4-1)\ldots(2^{2n-2}-1)2^{n-1}};$$

where n is the number of the term in the series.

We have Euler's formula

$$\text{Sin}\,\theta = 2^n \sin 2^{-n}\theta \cos\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta\ldots\cos 2^{-n}\theta.$$

When $n=\infty$ this becomes

$$\begin{aligned}\text{Sin}\,\theta &= \theta\cos\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta\ldots \textit{ad. inf.}\\ &= \theta(1-2\sin^2\tfrac{1}{4}\theta)(1-2\sin^2\tfrac{1}{8}\theta)\ldots.\end{aligned}$$

Substituting circular measures for sines, we have

$$\text{Sin } \theta > \theta\,(1 - 2^{-3}\theta^2)\,(1 - 2^{-5}\theta^2)\,(1 - 2^{-7}\theta^2) \ldots \textit{ad. inf.}$$

Expanding this binomial product and arranging according to powers of θ, we observe that the coefficient of the nth term is a set of geometrical series whose terms are the reciprocal powers of two, the exponents being respectively the sums of the numbers 3, 5, 7, ... taken $n - 1$ at a time.

Thus, the coefficient of θ^3 is

$$2^{-3} + 2^{-5} + \ldots = \frac{1}{(2^2 - 1)\,2^1}\,.$$

That of θ^5 is

$$\left.\begin{array}{l} 2^{-8} \;\; + 2^{-10} + \ldots \\ + 2^{-12} + 2^{-14} + \ldots \\ \ldots\ldots\ldots\ldots \end{array}\right\} = \frac{1}{(2^2 - 1)\,(2^4 - 1)\,2^2}\,.$$

That of θ^7,

$$\left.\begin{array}{l} (2^{-15} + 2^{-17} + \ldots) + (2^{-19} + 2^{-21} + \ldots) + \ldots \\ (2^{-21} + 2^{-23} + \ldots) + (2^{-25} + 2^{-27} + \ldots) + \ldots \\ (2^{-27} + 2^{-29} + \ldots) + (2^{-31} + 2^{-33} + \ldots) + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} = \frac{1}{(2^2 - 1)\,(2^4 - 1)\,(2^6 - 1)\,2^3}\,.$$

Thus, the general form of the series is easily recognized; and we have, finally,

$$\text{Sin } \theta > \theta - \frac{\theta^3}{(2^2 - 1)\,2^1} + \frac{\theta^5}{(2^2 - 1)\,(2^4 - 1)\,2^2} - \ldots$$

$$+ (-1)^{n+1} \frac{\theta^{2n-1}}{(2^2 - 1)\,(2^4 - 1) \ldots (2^{2n-2} - 1)\,2^{n-1}} \pm \ldots$$

[*W. H. Echols.*]